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Nonexistence of Riemann Solutions and Majda–Pego Instability

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We study Riemann problems for the shallow water equations. We consider weak self-similar Riemann solutions consisting of constant states, rarefaction waves, and/or jump discontinuities that satisfy the viscous profile entropy criterion, with a positive definite, symmetric viscosity matrix. We prove that for a "generic" symmetric, positive definite viscosity matrix there is an open set of Riemann initial data for which a weak self-similar Riemann solution does not exist. We show that this happens for the hyperbolic initial data that is unstable in the sense studied by Majda and Pego. We prove that such initial data always exist for positive definite, symmetric, nondiagonal viscosity matrices. In the work that follows previous work by the authors (in press, *Nonlinear Anal.*) we show that in the situations presented in this paper, measure-value solutions exhibiting continuously generated oscillations take place. The results of the present paper provide a new insight into the role of the viscous profile entropy criterion and the Majda–Pego instability in the existence of Riemann solutions for nonlinear conservation laws. © 2001 Academic Press

1. INTRODUCTION

In this work we study Riemann problems for the shallow water equations of the form [14]

$$U_t + F(U)_x = 0,$$

where

$$U = \begin{bmatrix} v \\ \phi \end{bmatrix} \qquad \text{and} \qquad F(U) = \begin{bmatrix} v^2/2 + \phi \\ v\phi \end{bmatrix}.$$

We consider weak self-similar solutions that consist of constant states, rarefaction waves and/or jump discontinuities that satisfy the viscous profile

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entropy criterion with a general symmetric, positive definite viscosity matrix. Although many of the Riemann solutions' features appear when a multiple of the identity matrix is used, this is a nongeneric situation and, to capture physically relevant solutions, viscosity matrices other than the identity need to be considered.

The goal of this work is to investigate whether there are symmetric, positive definite viscosity matrices and hyperbolic Riemann initial data that lead to nonexistence of weak self-similar solutions. Indeed, we show that for a "generic" symmetric, positive definite viscosity matrix there exists an open set of Riemann initial data for which we prove that a weak self-similar solution does not exist. Furthermore, we associate this behavior with the presence of the, so called, Majda–Pego unstable region of hyperbolic states, studied in [2, 15]. We show that a nontrivial Majda–Pego unstable region always exists for every symmetric, nondiagonal, positive definite viscosity matrix, and that nonexistence of Riemann solutions will always be present in those situations.

The behavior studied in this work is not isolated to the shallow water equations. In [1] nonexistence of a weak self-similar solution was established for a three-phase flow model arising in oil reservoir modeling. The viscosity matrix in the three-phase flow model derives from capillary pressures and is symmetric, nondiagonal, positive definite. The nonexistence result for the three-phase flow equations was obtained as a combination of analytical methods and numerical simulations. In contrast with the shallow water equations, the three-phase flow model admits nonclassical (transitional) waves and construction of Riemann solutions was impossible without the help of numerical simulations. In the present paper we were able to obtain nonexistence of weak Riemann solutions by using only analytical methods. We prove that the nonexistence result holds for a general symmetric, positive definite matrix and an open set of Riemann initial data. To our knowledge, this is the first time that nonexistence of Riemann solutions has been proved with Riemann data in the strictly hyperbolic region and with the viscous entropy criterion employed to select physically meaningful shock waves.

To show the main result of this work we consider initial data that correspond to a Lax admissible shock wave that does not possess a viscous profile due to the presence of a limit cycle in the dynamical system associated with the viscous profile entropy criterion. We identify such shock waves by studying the universal unfolding of a Bogdanov–Takens bifurcation that takes place at the boundary of the region in state space consisting of points that are stable in the sense studied by Majda and Pego [15]. The initial shock wave data correspond to the critical points in the dynamical systems that belong to the limit cycle region in the universal unfolding of the Bogdanov–Takens bifurcation [10]. For such initial data we establish nonexistence of a weak Riemann solution by showing that it is impossible to use other waves arising in this model to solve the Riemann problem.

Existence of viscous profiles has been studied by various authors [3, 4, 6, 16]. Most works assume genuine nonlinearity or $D(U)=I$. An exception is a work by Mock [16] in which a broad class of viscosity matrices is identified that ensures existence of connecting orbits for compressive shock waves in the large. Conley and Smoller [3] presented an example of a constant, symmetric, positive definite matrix which is inadmissible for the p-system of the isentropic gas dynamics equations in the sense that for any Riemann initial data, standing wave solutions of the associated dynamical system cannot converge to any shock wave solution of that Riemann problem. This is related to the present work in that the shallow water equations are equivalent to the isentropic equations of gas dynamics with $\gamma=2$. Therefore, it is reasonable to expect nonexistence to occur in gas dynamics as well.

To understand what kind of a solution exists in the situations studied in this work, in [17] we studied solutions of the inviscid conservation law obtained as a singular limit of the viscous regularizations. We showed that the solutions that exist in the situations when a weak self-similar solution does not exist, satisfy the system of conservation laws in a measure-valued sense, and exhibit continuously generated oscillations that increase in frequency, but stay uniformly bounded. Such solutions have already been observed, for example, in models for phase transitions, in particular, the formation of a band-like structure in a crystal of memory alloy during austenitic-martensitic transformation. In the form of a conservation law oscillatory solutions for phase transitions were studied in [7]. Oscillations of this type have also been observed by Majda and DiPerna in incompressible flow [5]. Whether the oscillatory solutions are physical or not is still an open question. The answer very likely depends on the model at hand.

The main results of this paper are:

(1) the proof of nonexistence of a weak self-similar solution for an open set of model parameters, and

(2) a study of the relationship between nonexistence of weak Riemann solutions and the presence of a nontrivial Majda–Pego region.

We present these results in three sections. We first describe the model equations and Riemann initial data in Section 2. In Section 2 we also specify a complete list of admissible waves that can be used in this model to construct a weak self-similar solution. In Section 3 we study the region of Majda–Pego unstable points. We show that a nontrivial Majda–Pego unstable region occurs for every symmetric, nondiagonal, positive definite matrix. At the boundary of the Majda–Pego region a Bogdanov–Takens bifurcation takes place. We calculate the universal unfolding of the Bogdanov–Takens bifurcation and show that there is a region in the two-dimensional parameter plane that consists of points for which the associated dynamical system has a limit cycle surrounding one of the critical points. The dynamical system is derived from the viscous profile entropy criterion. The Riemann initial data (U_L, U_R) that we consider in this paper correspond to the critical points that arise in the universal unfolding of a Bogdanov–Takens bifurcation and have the property that either U_L or U_R is surrounded by a limit cycle. In addition, U_L and U_R satisfy the Lax characteristic inequalities. The presence of a limit cycle prevents a shock wave from U_L to U_R from possessing a viscous profile. In Section 4 we show that for such initial data it is impossible to construct a Riemann solution using the waves at our disposal (the rarefaction waves and viscous profilable shock waves).

2. DESCRIPTION OF THE PROBLEM

2.1. *The Model Equations*

We study the shallow water equations [14]

$$\begin{bmatrix} v \\ \phi \end{bmatrix}_t + \begin{bmatrix} v^2/2+\phi \\ v\phi \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{1}$$

where v is the velocity and ϕ is the depth of water times a constant gravitational acceleration. This is a strictly hyperbolic system of conservation laws for $\phi > 0$ since the eigenvalues of the Jacobian of the flux function

$$F'(U) = \begin{bmatrix} v & 1 \\ \phi & v \end{bmatrix},$$

$\lambda_1 = v - \sqrt{\phi}$ and $\lambda_2 = v + \sqrt{\phi}$, are real and distinct for $\phi > 0$. The *coincidence line*, i.e., the line along which the eigenvalues of the Jacobian of the flux function coincide, corresponds to the zero depth, $\phi = 0$. We consider Riemann initial data for this system of conservation laws, namely, the initial data that is piecewise constant, separated by a jump discontinuity

$$U(x,0) = \begin{cases} U_L, & x < 0 \\ U_R, & x > 0. \end{cases} \tag{2}$$

A weak solution of a system of conservation laws with initial data $U(x,0)$ is a bounded measurable function $U(x,t)$ such that

$$\int_0^\infty \int_{-\infty}^\infty U\phi_t + F(U)\,\phi_x \,dx\,dt + \int_{-\infty}^\infty U(x,0)\,\phi(x,0)\,dx = 0, \tag{3}$$

for all $\phi(x, t) \in C_0^\infty$. This weak formulation of the conservation law implies that in order for a jump discontinuity between two states U_- and U_+, traveling with speed s, to be a weak solution, the following condition, called the Rankine-Hugoniot condition, must hold

$$-s[U_+ - U_-] + F(U_+) - F(U_-) = 0. \tag{4}$$

Because there are solutions of (4) that are not physically meaningful, additional criteria need to be imposed to extract the physical solution. Most commonly used are the Lax characteristic entropy condition [13] and the viscous profile entropy condition [9]. In this paper we consider weak self-similar solutions of the Riemann initial-value problem, consisting of constant states, rarefaction waves and/or jump discontinuities (shock waves) that satisfy the viscous profile entropy criterion, with a positive definite viscosity matrix of the form

$$D = \begin{bmatrix} a & -b \\ -b & c \end{bmatrix}. \tag{5}$$

We consider Riemann data (2) with the property that U_L and U_R form a Lax-admissible shock wave [13] that does not admit a viscous profile due to the presence of a limit cycle in the associated dynamical system. We describe the admissibility criterion and the dynamical system below.

We demonstrate the general results obtained throughout this paper on a particular example of the Riemann problem for the shallow water equations (1) with the following choices of the Riemann initial data and of the viscosity parameters:

$$U_L = \begin{pmatrix} 0.00 \\ 0.12 \end{pmatrix},$$

$$U_R = \begin{pmatrix} -0.1840 \\ 0.0642 \end{pmatrix}, \qquad D = \begin{pmatrix} 13.0116 & -5.9144 \\ -5.9144 & 3.1429 \end{pmatrix}. \tag{6}$$

These parameters are convenient for the numerical simulation of the solution of this Riemann problem, obtained in [17]. In [17] we apply the general results obtained in the present paper to the Riemann problem specified above, to conclude that there is no weak solution consisting of constant states, rarefaction waves and/or shock waves that satisfy the viscous profile entropy criterion with the viscosity matrix specified in (6). We then calculate the solution of this Riemann problem by considering the inviscid limit of the solutions of the associated parabolic problems (7). The solutions of the parabolic problems exhibit continuously generated oscillations that increase in frequency and stay uniformly bounded in amplitude as $\varepsilon \to 0$. We prove that they converge, in the weak-* topology of L^∞, to a measure-valued solution of the conservation law.

2.2. *Admissibility of Shock Waves: Viscous Profile Entropy Criterion*

We consider a shock wave *admissible* if it satisfies the *viscous profile entropy criterion* [9] with a positive definite viscosity matrix given in (5). A shock between U_- and U_+ traveling with speed s, (U_-, U_+, s), is considered admissible if it can be obtained in the limit, as $\varepsilon \to 0$, of traveling wave solutions $\tilde{U}(\xi) = \tilde{U}((x-st)/\varepsilon)$ of the parabolic system

$$U_t + F(U)_x = \varepsilon(D(U)\, U_x)_x, \tag{7}$$

with the asymptotic states $\lim_{\xi \to \pm\infty} \tilde{U} = U_\pm$. Such a traveling wave corresponds to a connecting orbit in the dynamical system

$$U' = D^{-1}[-s(U-U_-) + F(U) - F(U_-)] \tag{8}$$

between the critical points U_- and U_+, traversed in the direction from U_- to U_+. A complete list of admissible shock waves that may comprise a weak Riemann solution is the following:

- a compressive repellor $\to$ saddle shock wave, or a 1-shock wave (which corresponds to a connecting orbit from a repellor U_- to a saddle U_+, with the characteristic structure consisting of three converging characteristics and one outgoing characteristic),
- a compressive saddle $\to$ attractor shock wave, or a 2-shock wave (which corresponds to a connecting orbit from a saddle U_- to an attractor U_+, with the characteristic structure consisting of three converging characteristics and one outgoing characteristic),
- a saddle $\to$ saddle shock wave, or a transitional shock wave (which corresponds to a connecting orbit from a saddle U_- to a saddle U_+; thus the characteristic structure consists of two converging and two outgoing characteristics),
- a repellor $\to$ attractor shock wave, or an overcompressive shock wave (which corresponds to a connecting orbit from a repellor U_- to an attractor U_+; thus the characteristic structure consists of four converging characteristics).

We note that if D is different from a multiple of the identity, a critical point that is a repellor in the case when $D = I$ may become an attractor for $D \neq I$, and similarly, an attractor for $D = I$ may become a repellor for $D \neq I$. (Since D is positive definite, saddle points for $D = I$ stay saddle points for $D \neq I$.) One consequence of this property is that there are repellor to saddle shock waves and saddle to attractor shock waves whose characteristic structure consists of three outgoing and one incoming characteristic. These expansive or rarefaction-shock waves have been studied in [3]. Their presence is tied with the existence of a nontrivial "Majda–Pego-region of linearized instability," presented in Subsection 3.1. Based on the long term stability considerations we disregard expansive shock waves. This is because expansive shock waves were shown to be unstable under small perturbations as solutions of the parabolic PDE (7). See [18]. This is because the Majda–Pego instability occurs in the outgoing characteristic field and thus can never get swallowed by the shock.

In the following section we describe the Majda–Pego region, calculate the universal unfolding of a Bogdanov–Takens bifurcation occurring at the boundary of the Majda–Pego region, and discuss the Riemann initial data considered in this work.

3. MAJDA–PEGO REGION AND THE BOGDANOV–TAKENS UNFOLDING

3.1. *Majda–Pego Region*

In [15] Majda and Pego studied the criteria for a viscosity matrix D to be strictly stable in the sense that for every state U, the Cauchy problem for the linearized equation

$$V_t + F'(U)\, V_x = \varepsilon D(U)\, V_{xx}$$

is uniformly well-posed in L^2 as $\varepsilon \to 0$. For a 2×2 system they showed that the stable viscosity matrix criterion is

$$l_j(U)\, D(U)\, r_j(U) \geqslant 0, \qquad j = 1, 2,$$

where $l_j(U)$ and $r_j(U)$ are the left and the right eigenvectors of $F'(U)$. In [2] Čanić and Plohr viewed this condition as a condition on the state U being stable or not in the sense studied by Majda and Pego. Stable points are called the Majda–Pego points. The following result from [2] will be used below.

PROPOSITION 1. *A state U is a Majda–Pego point for the viscosity matrix D if and only if $tr[-s + F'(U)]$ and $tr(D^{-1}[-s + F'(U)])$ have the same (nonzero) sign for each eigenvalue s of $F'(U)$.*

Using this proposition we calculate the Majda–Pego boundary for the model in consideration. For the first eigenvalue we have

$$[-\lambda_1(U)+F'(U)]=\begin{bmatrix}\sqrt{\phi} & 1\\ \phi & \sqrt{\phi}\end{bmatrix}$$

and

$$D^{-1}[-\lambda_1(U)+F'(U)]=\frac{1}{ac-b^2}\begin{bmatrix}c & b\\ b & a\end{bmatrix}\begin{bmatrix}\sqrt{\phi} & 1\\ \phi & \sqrt{\phi}\end{bmatrix},$$

and for the second

$$[-\lambda_2(U)+F'(U)]=\begin{bmatrix}-\sqrt{\phi} & 1\\ \phi & -\sqrt{\phi}\end{bmatrix}$$

and

$$D^{-1}[-\lambda_2(U)+F'(U)]=\frac{1}{ac-b^2}\begin{bmatrix}c & b\\ b & a\end{bmatrix}\begin{bmatrix}-\sqrt{\phi} & 1\\ \phi & -\sqrt{\phi}\end{bmatrix}.$$

For U to be a Majda–Pego point we must have that

$$d_{11}\equiv\frac{tr(D^{-1}[-\lambda_1(U)+F'(U)])}{tr[-\lambda_1(U)+F'(U)]}$$

and

$$d_{22}\equiv\frac{tr(D^{-1}[-\lambda_2(U)+F'(U)])}{tr[-\lambda_2(U)+F'(U)]}$$

have the same positive sign. In our case

$$d_{11}=\frac{(c+a)\sqrt{\phi}+b(1+\phi)}{2\sqrt{\phi}}\qquad\text{and}\qquad d_{22}=\frac{-(c+a)\sqrt{\phi}+b(1+\phi)}{-2\sqrt{\phi}}.$$

THEOREM 1. *The region of Majda–Pego stable points consists of all points* (v,ϕ) *with* $\phi>0$ *such that*

$$(c+a)\sqrt{\phi}+b(1+\phi)>0 \tag{9}$$

and

$$-(c+a)\sqrt{\phi}+b(1+\phi)<0. \tag{10}$$

The only situation in which the entire state space $\phi>0$ is stable in the sense of Majda and Pego is when D is a diagonal, positive definite matrix. In all other cases, namely when $b\neq 0$, there is a nontrivial subregion of the state space consisting of the unstable Majda–Pego points.

Proof. From the definitions of d_{11} and d_{22} we see that in order for a point (v,ϕ) to be Majda–Pego stable we need that $(c+a)\sqrt{\phi}+b(1+\phi)>0$ and $-(c+a)\sqrt{\phi}+b(1+\phi)<0$.

If $b=0$ and D is positive definite we have that both inequalities are satisfied, and so the entire state space is stable.

If $b\neq 0$ there are two cases to consider.

Case 1. Let $b>0$. We have two subcases, depending on whether the discriminant $(a+c)^2-4b^2$ of the equations $(c+a)\sqrt{\phi}+b(1+\phi)=0$ and $-(c+a)\sqrt{\phi}+b(1+\phi)=0$ is positive or negative. Let $a+c>2b$ (for example, D is diagonally dominant). Then the zeros of the first equation are given by

$$\sqrt{\phi}=(-(c+a)\pm\sqrt{(c+a)^2-4b^2})/(2b).$$

Since $a+c>2b>0$ the two zeros are both negative, and since $b>0$, the corresponding parabola in $\sqrt{\phi}$ is positive for all $\phi>0$. Thus, the first inequality (9) is always satisfied. The second inequality (10) is satisfied for all the points ϕ inside the interval (ϕ_1^*,ϕ_2^*) where $\phi_1^*=(c+a-\sqrt{(c+a)^2-4b^2})^2/4b^2$ and $\phi_2^*=(c+a+\sqrt{(c+a)^2-4b^2})^2/4b^2$. Since $a+c>2b>0$ both ϕ_1^* and ϕ_2^* are positive. Therefore, there is a nontrivial (unbounded) region of the state space containing the unstable Majda–Pego points.

Now assume that $a+c<2b$. Then the discriminant of both equations is negative and since $b>0$, the first inequality is always satisfied but the second inequality is never true. Thus in this case the entire hyperbolic state space is Majda–Pego unstable.

Case 2. Let $b<0$. Then $a+c>2b$ since D is positive definite (namely, a and c both need to be positive). In this case the discriminant of both equations $(c+a)\sqrt{\phi}+b(1+\phi)=0$ and $-(c+a)\sqrt{\phi}+b(1+\phi)=0$ is positive, and since $b<0$, we have that $(c+a)\sqrt{\phi}+b(1+\phi)<0$ for all $\phi>0$ such that $\sqrt{\phi}>(-(c+a)+\sqrt{(c+a)^2-4b^2})/2b$. Thus, there is an unbounded region of state space that is Majda–Pego unstable. ∎

Assume that $a, b, c > 0$ and that $a + c > 2b$. Then the region of Majda–Pego stable points is defined by

$$\frac{1}{2b}\left(c + a - \sqrt{(c+a)^2 - 4b^2}\right) \leqslant \sqrt{\phi} \leqslant \frac{1}{2b}\left(c + a + \sqrt{(c+a)^2 - 4b^2}\right). \tag{11}$$

Notice that both bounds are strictly greater than zero. Figure 1 shows the Majda–Pego stable region for the parameters $a = 13.0116$, $b = 5.9144$, and $c = 3.1429$.

In the rest of the paper, without loss of generality, we will be assuming that D is positive definite $(a, c > 0, ac - b^2 > 0)$ and that D is diagonally dominant in the sense that $a + c > 2b > 0$.

In the following subsection we will study the relationship between the boundary of the Majda–Pego region and the Bogdanov–Takens bifurcation.

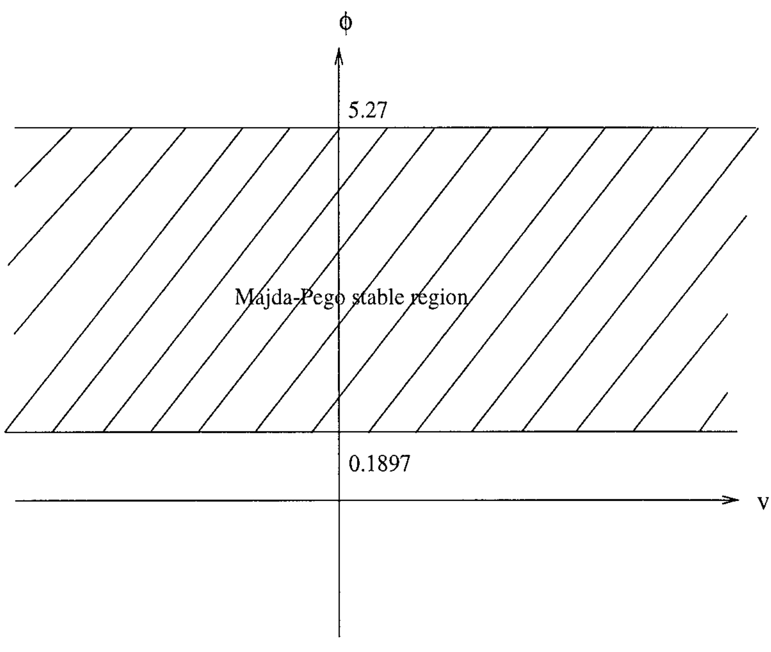

FIG. 1. The Majda–Pego stable region for the viscosity matrix (2) with $a = 13.0116$, $b = 5.9144$, and $c = 3.1429$. The figure shows that the region of stable points lies between $\phi = 0.1897$ and $\phi = 2.57$.

3.2. *The Unfolding of the Bogdanov–Takens Bifurcation*

Dynamical system (8) undergoes a Bogdanov–Takens bifurcation if one of the critical points U_{cr} of the dynamical system satisfies

$$\begin{aligned} \det\{D^{-1}[-s+F'(U_{cr})]\} &= 0, \\ \operatorname{tr}\{D^{-1}[-s+F'(U_{cr})]\} &= 0. \end{aligned} \tag{12}$$

Thus the Jacobian at U_{cr} is nilpotent. The set of all the points U_{cr} such that dynamical system (8) undergoes a Bogdanov–Takens bifurcation at $U_L = U_R = U_{cr} = U_-$ will be called the *Bogdanov–Takens locus*.

The universal unfolding of the Bogdanov–Takens bifurcation can be found, for example, in [10]. The universal unfolding is a two-parameter unfolding, containing the Hopf locus and the homoclinic locus. The *Hopf locus* comprises points that correspond to dynamical systems that contain a nonhyperbolic critical point U_{cr} for which the discriminant of the Jacobian is negative and the trace of the Jacobian vanishes:

$$\begin{aligned} \operatorname{discrm}\{D^{-1}[-s+F'(U_{cr})]\} &< 0, \\ \operatorname{tr}\{D^{-1}[-s+F'(U_{cr})]\} &= 0. \end{aligned} \tag{13}$$

The *homoclinic locus* comprises points that correspond to dynamical systems for which the homoclinic loop bifurcation takes place. In contrast to the Hopf locus, the existence of a homoclinic orbit is a nonlocal phenomenon. Although we do not know an explicit formula for the homoclinic locus, the tangent space of this locus can be calculated.

To obtain the main result of this paper we will use the following proposition, proved in [2].

PROPOSITION 2. *The boundary of the set of Majda–Pego points is contained in the union of the coincidence locus and the Bogdanov–Takens locus* (*occurring for* $U_L = U_R$).

At the boundary of the Majda–Pego region that corresponds to the Bogdanov–Takens locus, we will calculate the universal unfolding of the Bogdanov–Takens bifurcation. We will find the Hopf locus and the tangent to the homoclinic locus and show that there is a nontrivial region of points between these two loci whose dynamical systems contain a limit cycle surrounding one of the states U_L or U_R in the initial data. This will prevent the existence of a connecting orbit from U_L to U_R and thus make the initial shock wave inadmissible. In Section 4 we will show that it is impossible to use other waves to construct a solution between U_L and U_R for such initial data.

The dynamical system associated with the shallow water equations and a fixed viscosity matrix of the form (5) is given by

$$\begin{bmatrix} \dot{v} \\ \dot{\phi} \end{bmatrix} = D^{-1} \begin{bmatrix} -s(v-v_-)+\phi-\phi_- + \frac{1}{2}(v+v_-)(v-v_-) \\ -s(\phi-\phi_-)+v(\phi-\phi_-)+\phi_-(v-v_-) \end{bmatrix}. \tag{14}$$

Let $x = v - v_-$ and $y = \phi - \phi_-$. The above system is equivalent to

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = D^{-1} \begin{bmatrix} -sx + y + \frac{1}{2}(v+v_-)\,x \\ -sy + vy + \phi_- x \end{bmatrix}.$$

Using $v - s = x - (s - \lambda_2(U_-)) + v_- - \lambda_2(U_-)$, and denoting

$$\mu = s - \lambda_2(U_-)$$

we obtain

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = D^{-1} \begin{bmatrix} (-\mu - \sqrt{\phi_-})\,x + y + \frac{1}{2}x^2 \\ \phi_- x + (-\mu - \sqrt{\phi_-})\,y + xy \end{bmatrix}.$$

Taking into account the form of the viscosity matrix we get

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \frac{1}{\det D} \begin{bmatrix} (c(-\mu-\sqrt{\phi_-})+b\phi_-)\,x + (c+b(-\mu-\sqrt{\phi_-}))\,y \\ (b(-\mu-\sqrt{\phi_-})+a\phi_-)\,x + (b+a(-\mu-\sqrt{\phi_-}))\,y \end{bmatrix} + Q(x, y), \tag{15}$$

where

$$Q(x, y) = \frac{1}{\det D} \begin{bmatrix} \frac{1}{2}cx^2 + bxy \\ \frac{1}{2}bx^2 + axy \end{bmatrix}.$$

This is a two-parameter family of dynamical systems in parameters μ and ϕ_-. We will denote the vector field on the right-hand side by $X(x, y; \mu, \phi_-)$, and its Jacobian by $X'(x, y; \mu, \phi_-)$. A simple calculation shows that there are generically three critical points of this vector field. They are given by $(0, 0)$, (x_1, y_1), and (x_2, y_2) where

$$x_i = \tfrac{1}{2}\left(3(\mu+\sqrt{\phi_-}) \pm \sqrt{\mu^2 + 2\mu\sqrt{\phi_-} + 9\phi_-}\right), \qquad i = 1, 2$$

$$y_i = \mu(\mu + 2\sqrt{\phi_-}) - \tfrac{5}{2}(\mu + \sqrt{\phi_-})\,x_i, \qquad i = 1, 2.$$

The critical point (x_1, y_1) has the property that it belongs to the elliptic region, namely, the value of y_1 is always less than $-\phi_-$ which means that the corresponding ϕ is negative. Thus there are only two critical points that belong to the hyperbolic region of the state space. This will be used in Section 4.

Vector field (15) undergoes a Bogdanov–Takens bifurcation at $x=0$, $y=0$ for the parameter values

$$\mu=0, \qquad \phi_-=\phi_-^*=\left[\frac{1}{2b}\left(c+a-\sqrt{(c+a)^2-4b^2}\right)\right]^2. \tag{16}$$

Indeed, at $(0, \phi_-^*)$ the Jacobian $X'(0, 0; 0, \phi_-^*)$ is given by

$$X'(0, 0; 0, \phi_-^*)=\begin{bmatrix} -c\sqrt{\phi_-^*}+b\phi_-^* & c-b\sqrt{\phi_-^*} \\ -b\sqrt{\phi_-^*}+a\phi_-^* & b-a\sqrt{\phi_-^*} \end{bmatrix}.$$

This matrix is of the form

$$X'(0, 0; 0, \phi_-^*)=\begin{bmatrix} \alpha & \beta \\ -\alpha^2/\beta & -\alpha \end{bmatrix},$$

where

$$\alpha=\frac{1}{2b}\left(ac+a^2-a\sqrt{(a+c)^2-4b^2}\right)-b, \tag{17}$$

and

$$\beta=\tfrac{1}{2}\left(c-a+\sqrt{(a+c)^2-4b^2}\right). \tag{18}$$

Its trace and the determinant are both equal to zero, which defines a Bogdanov–Takens bifurcation. If we employ a transformation by the non-singular matrix

$$R=\begin{bmatrix} \alpha & 0 \\ -\alpha^2/\beta & 1 \end{bmatrix}$$

we obtain the Jordan form

$$R^{-1}X'(0, 0; 0, \phi_-^*)\,R=\begin{bmatrix} 0 & \dfrac{\beta}{\alpha} \\ 0 & 0 \end{bmatrix}.$$

For R to be nonsingular we need $\alpha \neq 0$ and for the existence of the Jordan form we need $\beta \neq 0$. We now calculate the Hopf locus and the tangent to the homoclinic locus in the universal unfolding of the Bogdanov–Takens bifurcation taking place at $\mu = 0$, $\phi_- = \phi_-^*$.

Hopf Bifurcation Locus. The Hopf locus consists of points (μ, ϕ_-) such that the corresponding dynamical system has the property that at a non-hyperbolic critical point U_{cr} the following holds

$$\operatorname{discrm}\{D^{-1}[-s + F'(U_{cr})]\} < 0$$

and

$$\operatorname{tr}\{D^{-1}[-s + F'(U_{cr})]\} = 0.$$

We are interested in the Hopf bifurcation taking place at $U_{cr} = U_-$ which corresponds to $x = y = 0$. The trace of the Jacobian $X'(x, y; \mu, \phi_-)$ evaluated at $(x, y) = (0, 0)$ is given by

$$\operatorname{tr}(X'(0, 0; \mu, \phi_-)) = \frac{1}{\det D}((a + c)(-\mu - \sqrt{\phi_-}) + b(1 + \phi_-)),$$

where $\det D = ac - b^2$ is the determinant of the viscosity matrix D. Thus, the Hopf locus is defined by

$$(a + c)(-\mu - \sqrt{\phi_-}) + b(1 + \phi_-) = 0 \tag{19}$$

in the region where the discriminant is negative. The discriminant of $X'(0, 0; \mu, \phi_-)$ is equal to $\operatorname{tr}^2(X'(0, 0; \mu, \phi_-)) - 4 \det(X'(0, 0; \mu, \phi_-))$ where the determinant $\det(X'(0, 0; \mu, \phi_-))$ of the Jacobian evaluated at zero is given by

$$\det(X'(0, 0; \mu, \phi_-)) = (ac - b^2)\, \mu(\mu + 2\sqrt{\phi_-}).$$

For the parameters $a = 13.0116$, $b = 5.9144$ and $c = 3.1429$ the Hopf locus is shown in Fig. 2.

Homoclinic Bifurcation Locus. We use the Melnikov's integral analysis (see, for example, [10]), to calculate the locus tangent to the homoclinic bifurcation locus.

We first write the system (15) as a perturbation in μ and ϕ_- around the Bogdanov–Takens bifurcation point $\mu = 0$, $\phi_- = \phi_-^*$ determined by the left Majda–Pego boundary given in (11).

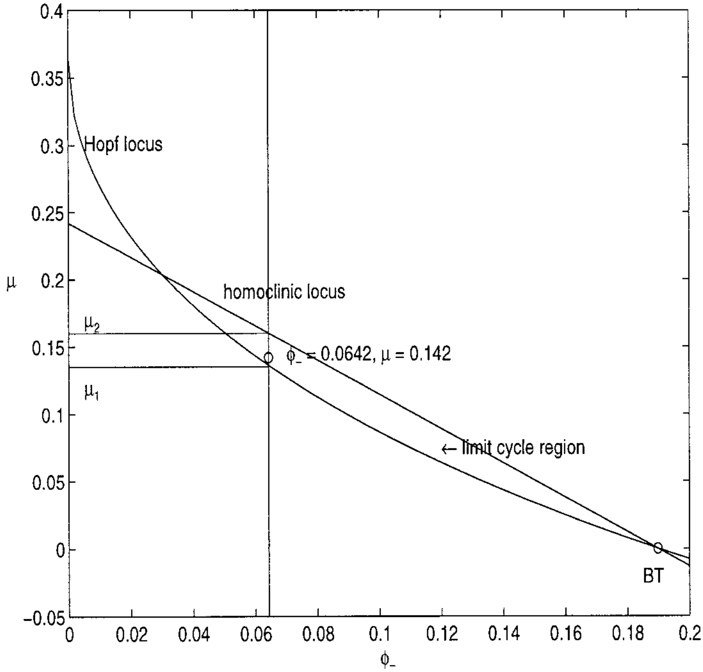

FIG. 2. Universal unfolding of the Bogdanov–Takens bifurcation for the shallow water equations. The figure shows the Hopf and the homoclinic bifurcation loci in the unfolding of the Bogdanov–Takens bifurcation calculated for the viscosity parameters $a = 13.0116$, $b = 5.9114$ and $c = 3.1429$. The Bogdanov–Takens bifurcation, denoted by BT, occurs at the boundary of the Majda–Pego region determined by $\phi^*_- = 0.1897$. The region between the Hopf and the homoclinic locus contains the points for which the corresponding dynamical system has a limit cycle. One such point, $(\phi_-, \mu) = (0.0642, 0.142)$, is shown in this figure. The associated phase space portrait showing the limit cycle is depicted in Fig. 3. The points on the vertical line $\phi_- = 0.0642$ define dynamical systems along the Hugoniot curves (2-branch) through (v_-, ϕ_-) where $\phi_- = 0.0642$. Figure 4 shows a portion of such a Hugoniot curve for $(v_-, \phi_-) = (v_R, \phi_R) = (-0.1840, 0.0642)$.

We employ the transformation of coordinates

$$\begin{bmatrix} x \\ y \end{bmatrix} = R \begin{bmatrix} w \\ z \end{bmatrix}$$

and write system (15) as a perturbation around the Bogdanov–Takens point, using Taylor's expansion to approximate $\sqrt{\phi_-}$ around $\sqrt{\phi^*_-}$. We obtain the dynamical system of the form

$$\begin{bmatrix} \dot{w} \\ \dot{z} \end{bmatrix} = \frac{1}{\det D} \left(\begin{bmatrix} 0 & \frac{\beta}{\alpha} \\ 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \widetilde{\phi_-} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \right) \begin{bmatrix} w \\ z \end{bmatrix} + \tilde{Q}(w, z), \tag{20}$$

where

$$\tilde{Q}(w, z) = \frac{1}{\det D} \begin{bmatrix} w(\alpha_1 w + \beta_1 z) \\ w(\alpha_2 w + \beta_2 z) \end{bmatrix},$$

and

$$\widetilde{\phi_-} = \phi_- - \phi_-^* .$$

The coefficients a_{ij}, b_{ij}, α_i, and β_i are given in the Appendix.

Next we introduce a small parameter ε and use it to rescale the dynamical system so that for $\varepsilon = 0$ the dynamical system is Hamiltonian and has a homoclinic loop. The rescaling by ε "blows up" the Bogdanov–Takens critical point into a saddle and a spiral which is surrounded by a homoclinic loop to the saddle point obtained by the blow-up. The rescaling by ε is given by

$$w = \varepsilon^2 \bar{w}, \qquad z = \varepsilon^3 \bar{z}, \qquad \mu = \varepsilon^2 \bar{\mu}, \qquad \widetilde{\phi_-} = \varepsilon^2 \overline{\widetilde{\phi_-}}, \qquad t = (\det D)\, \tau/\varepsilon. \tag{21}$$

For simplicity, we will drop the bar over the new variables, and use the same notation further in the text. From this point on, only the rescaled variables appear and define the tangent to the homoclinic bifurcation locus. After rescaling we obtain a system of the form

$$\begin{bmatrix} \dot{w} \\ \dot{z} \end{bmatrix} = f(w, z) + \varepsilon g(w, z) + O(\varepsilon^2),$$

where

$$f(w, z) = \begin{bmatrix} \frac{\beta}{\alpha} z \\ (\mu a_{21} + \widetilde{\phi_-}\, b_{21})\, w + \alpha_2 w^2 \end{bmatrix}$$

and

$$g(w, z) = \begin{bmatrix} (\mu a_{11} + \widetilde{\phi_-}\, b_{11})\, w + \alpha_1 w^2 \\ (\mu a_{22} + \widetilde{\phi_-}\, b_{22})\, z + \beta w z \end{bmatrix}.$$

The Hamiltonian function defined by f is given by

$$H(w, z; \mu, \widetilde{\phi_-}) \equiv \frac{z^2}{2} + \gamma_1 \frac{w^2}{2} + \gamma_2 w^3,$$

where

$$\gamma_1 = \frac{\alpha}{\beta}(-a_{21}\mu - \widetilde{\phi_-}\, b_{21}), \qquad \gamma_2 = \frac{\alpha}{\beta}\left(\frac{-\alpha_2}{3}\right).$$

The homoclinic loop corresponds to the level curve $H(w, z; \mu, \widetilde{\phi_-}) = 0$. The Melnikov function [10] is defined by

$$M(\mu, \widetilde{\phi_-}) \equiv \int_{int\, H_0} \operatorname{div} g(w, z; \mu, \widetilde{\phi_-}),$$

where

$$H_0 \equiv \{(w, z) \mid H(w, z; \mu, \widetilde{\phi_-}) = 0\}.$$

It measures the separation of the stable and unstable manifolds on a cross-section transverse to the homoclinic orbit. The curve $M(\mu, \widetilde{\phi_-}) = 0$ determines the homoclinic manifold up to $O(\varepsilon^2)$ in the neighborhood of the Bogdanov–Takens bifurcation point. A calculation gives that the tangent locus is given by

$$7\gamma_2[\mu(a_{11} + a_{22}) + \widetilde{\phi_-}(b_{11} + b_{22})] + 2\gamma_1(2\alpha_1 + \beta_2) = 0. \tag{22}$$

Figure 2 shows the homoclinic locus obtained for the parameters $a = 13.0116$, $b = 5.9144$ and $c = 3.1429$.

Thus, we have proved the following theorem.

THEOREM 2. *Suppose that the coefficients of the matrix D are such that* $\alpha \neq 0$ *and* $\beta \neq 0$. *Furthermore, assume that the Bogdanov–Takens point* $(0, \phi_-^*)$ *is nondegenerate in the sense that* $\alpha_2 \neq 0$. *Then the Hopf bifurcation locus and a tangent to the homoclinic bifurcation locus that arise in the universal unfolding of the Bogdanov–Takens point are given, respectively, by*

$$(a + c)(-\mu - \sqrt{\phi_-}) + b(1 + \phi_-) = 0, \tag{23}$$

$$7\gamma_2[\mu(a_{11} + a_{22}) + (\phi_- - \phi_-^*)(b_{11} + b_{22})] + 2\gamma_1(2\alpha_1 + \beta_2) = 0 \tag{24}$$

in the region where the discriminant is negative.

The Bogdanov–Takens unfolding for the values $a=13.0116$, $b=5.9144$ and $c=3.1429$ is shown in Fig. 2. All the points that lie in the region between the Hopf and the homoclinic locus have the property that their corresponding dynamical systems have a limit cycle surrounding the critical point $U=U_-$. One such point is shown in Fig. 3. It corresponds to the values $\mu=0.142$, $\phi_-=0.0647$. The corresponding dynamical system is shown in Fig. 3. In the next section we will show that it is impossible to construct a Riemann solution with Riemann data corresponding to the limit cycle region.

We end this section by showing that the entire Hopf bifurcation locus consists of points (dynamical systems) for which the corresponding states (ϕ_-, v_-) are Majda–Pego unstable. More precisely, we will show that $\phi_- < \phi^*_-$, where $\phi^*_- = [\frac{1}{2b}(c+a-\sqrt{(c+a)^2-4b^2})]^2$ determines the lower boundary of Majda–Pego stable region, as defined in (11). Because the strict inequality holds, this implies that there exists a *neighborhood* of the Hopf locus consisting of points for which the corresponding states (ϕ_-, v_-) are Majda–Pego unstable.

It is still an open question whether both the left state U_L and the right state U_R in the initial data corresponding to the limit cycle region, are always Majda–Pego unstable or not. What can easily be seen is that in the example specified in (6), for which the corresponding parameter values are given by $\phi_-=0.0642$, $\mu=0.142$, as shown in Fig. 2, both states, U_L and U_R, are Majda–Pego unstable. This is because both $\phi_R=0.0642$ and

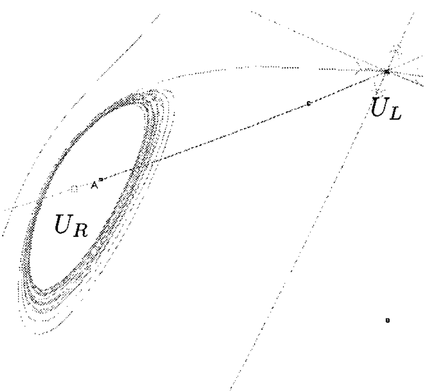

FIG. 3. The figure shows a portion of the phase space portrait corresponding to $(\phi_-, \mu) = (0.0642, 0.142)$ from Fig. 2. Shown is the state $U_L=(0, 0.12)$ which is a saddle point, and the state $U_R=U_-=(-0.1840, 0.0642)$ which is an attractor surrounded by a repelling limit cycle. Also shown in this figure is a part of the Hugoniot locus through U_R containing U_L. This figure was obtained using the Riemann solver package [9].

$\phi_L = 0.12$ are less than $\phi_-^* = 0.1897$ which determines the lower boundary of the Majda–Pego stable region, shown in Fig. 1.

We now show that the entire Hopf locus consists of points (dynamical systems) for which (ϕ_-, v_-) are Majda–Pego unstable. Recall that we are assuming that $a, c, b > 0$, $ac - b^2 > 0$ and $a + c > 2b$.

PROPOSITION 3. *The entire Hopf locus* (23) *arising in the universal unfolding of the Bogdanov–Takens bifurcation taking place at* $\mu = 0$, $\phi_- = \phi_-^*$, *defined by* (16), *consists of points for which the corresponding states* (ϕ_-, v_-) *are linearly unstable in the sense of Majda and Pego.*

Proof. Recall that Hopf bifurcation occurs along the locus (23) in the region where the discriminant is negative. At the Bogdanov–Takens bifurcation the discriminant is zero since both the trace and the determinant are equal to zero. A calculation of the discriminant along the Hopf locus gives

$$\operatorname{discrm}(X') = \frac{4[E\phi_-^2 + F\phi_- + E]}{(a+c)^2},$$

where $E = b^4 - b^2 ac$ and $F = 2b^4 - 4b^2 ac - b^2 c^2 - b^2 a^2 + a^3 c + 2a^2 c^2 + ac^3$. The discriminant is zero if and only if ϕ_- equals one of the roots of the quadratic polynomial in the numerator, which are

$$\{\phi_-\}_{1,2} = -1 + \frac{a+c}{b}\left[\frac{a + c \mp \sqrt{(a+c)^2 - 4b^2}}{2b}\right].$$

A simple calculation shows that $\{\phi_-\}_1 = [\sqrt{\phi_-^*}]^2$. Since D is positive definite, we have $b^2 - ac < 0$, and since $b \neq 0$, we have $E < 0$. This means that the discriminant is negative if and only if $\phi_- \in (-\infty, \{\phi_-\}_1) \cup (\{\phi_-\}_2, \infty)$. (Notice that the assumption $a + c > 2b$ implies that both roots are real.) Since the Bogdanov–Takens unfolding is calculated in a neighborhood of $\{\phi_-\}_1$, we consider the Hopf locus in the region where $\phi_- < \{\phi_-\}_1 = \phi_-^*$. By inequalities (2) this is a region that contains points (ϕ_-, v_-) that are *unstable* in the sense of Majda and Pego. This completes the proof.

4. NONEXISTENCE OF WEAK SOLUTIONS

In this section we discuss nonexistence of scale-invariant weak solutions consisting of constant states, rarefaction waves, and/or viscous admissible shock waves, for the class of Riemann problems described in the previous section.

The waves that could be used to construct a scale-invariant solution consisting of constant states, rarefaction waves and jump discontinuities are the 1- and 2-rarefaction waves, the viscous admissible compressive 1 and 2-shock waves, the composite waves (adjoint rarefaction waves and shock waves of the same family), the viscous admissible transitional shock waves, the transitional rarefaction waves, and the overcompressive shock waves. We will show that there cannot be transitional rarefaction waves, transitional shock waves, overcompressive waves or composite waves, which leaves only the "classical" waves: the compressive 1- and 2-shock waves, and the 1- and 2-rarefaction waves.

Transitional rarefaction waves occur when a 2-rarefaction wave is joined on its right by a 1-rarefaction wave. See [12]. At the point of connection, the eigenvalues of $F'(U)$ must coincide. Since

$$F'(U)=\begin{bmatrix} v & 1 \\ \phi & v \end{bmatrix}$$

the eigenvalues of $F'(U)$ are $\lambda_1=v-\sqrt{\phi}$ and $\lambda_2=v+\sqrt{\phi}$. Note that the strictly hyperbolic region is the open half-plane where $\phi>0$, and the eigenvalues coincide only at $\phi=0$. Hence for a transitional rarefaction wave to occur, λ_2 must be increasing as it approaches $\phi=0$. However, $\nabla\lambda_2\cdot\mathbf{n}>0$, where $\mathbf{n}=[0, 1]$ is the normal to the boundary of the elliptic region. So λ_2 is decreasing as it approaches the line of coincidence. Thus we have the following proposition.

PROPOSITION 4. *Transitional rarefaction waves do not occur in this model.*

The following lemma shows that neither transitional shock waves nor overcompressive shock waves occur in our model. Recall that in Section 3 we showed that the dynamical system associated with the viscous profile entropy criterion has at most two critical points that lie in the strictly hyperbolic region of the state space, $\phi>0$. We will show that for any positive definite viscosity matrix D one of these two critical points is always a saddle and the other an anti-saddle (a node or a spiral). For this we need the Rankine–Hugoniot conditions, which are

$$-s(v-v_-)+v^2/2+\phi-v_-^2/2-\phi_-=0, \tag{25}$$

and

$$-s(\phi-\phi_-)+v\phi-v_-\phi_-=0. \tag{26}$$

Along the Hugoniot curve through U_- that is tangent to the first eigenvector $r_1(U)$ of $F'(U)$ we have that

$$v = v_- - \frac{\phi - \phi_-}{\sqrt{\bar{\phi}}}, \qquad \text{and} \qquad s = v_- - \frac{\phi}{\sqrt{\bar{\phi}}}, \tag{27}$$

where $\bar{\phi} = \frac{1}{2}(\phi + \phi_-)$. Similarly, along the Hugoniot curve tangent to $r_2(U)$ we have

$$v = v_- + \frac{\phi - \phi_-}{\sqrt{\bar{\phi}}}, \qquad \text{and} \qquad s = v_- + \frac{\phi}{\sqrt{\bar{\phi}}}. \tag{28}$$

We will use these to show the following

LEMMA 1. *For any D positive definite, it is impossible to have either two saddle points or two anti-saddle points* (*nodes or spirals*) *as critical points in the dynamical system* (15) *that lie in the hyperbolic state space determined by $\phi > 0$.*

Proof. Let $X(U; s, U_-) = D^{-1}\{-s(U - U_-) + F(U) - F(U_-)\}$. If U_{cr} is a critical point that is a saddle we have that at U_{cr} $\det(X'(U_{cr}; s, U_-)) < 0$. Similarly, at an anti-saddle point U_{cr}, $\det(X'(U_{cr}; s, U_-)) > 0$. Since D is positive definite we have that the sign of $\det(X') = \text{sign}((v - s)^2 - \phi)$. We will show that, given $U_- = (v_-, \phi_-)$, there does not exist a (v, ϕ) satisfying the Rankine–Hugoniot equations, such that $((v_- s)^2 - \phi_-)((v - s)^2 - \phi) > 0$. This will imply that one of the critical points has to be a saddle and the other an anti-saddle.

Using Eqs. (27) and (28) we obtain that

$$(v - s)^2 = \frac{2\phi_-^2}{\phi + \phi_-} \qquad \text{and} \qquad (v_- - s)^2 = \frac{2\phi^2}{\phi + \phi_-}.$$

Assume that $(v - s)^2 - \phi < 0$ and $(v_- - s)^2 - \phi_- < 0$. This corresponds to having saddle points at (v_-, ϕ_-) and (v, ϕ). Taking into account that ϕ and ϕ_- are positive we obtain that

$$(v - s)^2 - \phi < 0 \Leftrightarrow 2\phi_-^2 - \phi\phi_- - \phi^2 < 0,$$

and

$$(v_- - s)^2 - \phi_- < 0 \Leftrightarrow 2\phi^2 - \phi_-\phi - \phi_-^2 < 0.$$

The first inequality is true if $\phi > \phi_- > 0$, and the second if $\phi_- > \phi > 0$ which leads to a contradiction. Similarly, if we assume that $(v-s)^2 - \phi > 0$ and $(v_- - s)^2 - \phi_- > 0$ we obtain that the first inequality holds if $\phi_- > \phi > 0$ and the second if $\phi > \phi_- > 0$, which is a contradiction. ▌

Thus, we have shown that neither transitional nor overcompressive shock waves occur in this model.

Wave Curves. To study solutions consisting of classical waves we use wave curve analysis. For this purpose we define the forward and the backward i-wave curves. We define an *i-wave curve* through a fixed state U_0 to be the set of all states U which can be connected to U_0 by a wave of the ith characteristic family (an i-rarefaction, an i-shock, or an i-composite wave). If U_0 is on the left of the i-wave and U is on the right, the i-wave curve is called the *forward i-wave curve*. Otherwise, we call it the *backward i-wave curve*. To construct a Riemann solution between two states U_L and U_R consisting of only 1- and 2-waves, we find the forward 1-wave curve through U_L and the backward 2-wave curve through U_R. If the two curves intersect at some point U_M, the Riemann solution consists of two waves: a 1-wave from U_L to the intermediate state U_M and a 2-wave from U_M to U_R.

In our model a simple calculation shows that the forward 1-wave curve through a state $U_0 = (v_0, \phi_0)$ consists of the 1-rarefaction part

$$v = v_0 - 2\sqrt{\phi} + 2\sqrt{\phi_0} \qquad \text{for} \quad \phi < \phi_0, \tag{29}$$

and the 1-shock part which is contained in

$$v = v_0 - \frac{\phi - \phi_0}{\sqrt{\phi}} \qquad \text{for} \quad \phi > \phi_0. \tag{30}$$

The curve specified in (30) contains the points $U = (v, \phi)$ that belong to the Hugoniot curve (27) through U_0 and satisfy the Lax characteristic entropy condition. A subset of this curve contains the points that are viscous admissible 1-shock waves. Only the part of the 1-shock curve that corresponds to the viscous admissible 1-shock waves belongs to the forward 1-wave curve through U_0.

Similarly, a backward 2-wave curve through $U_0 = (v_0, \phi_0)$ consists of the 2-rarefaction part

$$v = v_0 + 2\sqrt{\phi} - 2\sqrt{\phi_0} \qquad \text{for} \quad \phi < \phi_0, \tag{31}$$

and the 2-shock part which is contained in

$$v = v_0 + \frac{\phi - \phi_0}{\sqrt{\phi}} \qquad \text{for} \quad \phi > \phi_0 . \tag{32}$$

The curve specified in (32) contains the points $U = (v, \phi)$ that belong to the Hugoniot curve (28) through U_0 and satisfy the Lax characteristic entropy condition. A subset of this curve contains the points that are viscous admissible 2-shock waves. Only the part of the backward 2-shock curve that corresponds to the viscous admissible 2-shock waves belongs to the backward 2-wave curve through U_0.

This structure of the wave curves is global. In particular, note that the wave curves do not have any composite wave parts. If they did, the composite segment would have to start at the inflection point (see, for example, [8]). At this point $\nabla\lambda_i(U) \cdot \mathbf{r}_i(U) = 0$, where the $\mathbf{r}_i$'s are the eigenvectors of $F'(U)$. Since $\mathbf{r}_1 = [1, -\sqrt{\phi}]^T$, $\mathbf{r}_2 = [1, \sqrt{\phi}]^T$, $\lambda_1 = v - \sqrt{\phi}$, and $\lambda_2 = v - \sqrt{\phi}$, we have $\nabla\lambda_i \cdot \mathbf{r}_i = 3/2$. Therefore, $\nabla\lambda_i \cdot \mathbf{r}_i \neq 0$. Thus there are no composite waves with a shock on the right.

We also cannot have composite waves with a shock on the left, called composite waves sonic on the left. Composite waves sonic on the left occur only when the *i*-shock speed, s, equals λ_i at some point on the *i*-shock curve. Consider, for example, the shock branch of the 1-wave curve through U_0 (30) with $\phi > \phi_0$. Using s from the Rankine–Hugoniot equation (26) and $\lambda_1 = v - \sqrt{\phi}$, we see that $s = \lambda_1$ only when

$$v = v_0 + \frac{\sqrt{\phi}(\phi - \phi_0)}{\phi_0}. \tag{33}$$

Obviously, the curves in Eqs. (30) and (33) do not intersect for $\phi > \phi_0$ and so there are no composite waves sonic on the left.

Therefore, the only waves that can be used in the construction of a weak self-similar Riemann solution are the "classical" compressive 1- and 2-shock waves and the 1- and 2-rarefaction waves.

Nonexistence of Weak Solutions. To find a weak solution it is sufficient to construct the forward 1-wave curve through U_L and the backward 2-wave curve through U_R and verify whether there is an intersection of these two curves. By studying global geometry of these curves we show below that they do not intersect.

Recall that the initial data that we consider in this paper are the shock initial data, i.e., U_L and U_R satisfy the Rankine–Hugoniot conditions (and the Lax characteristic entropy criterion) but there is no viscous profile from U_L to U_R due to the presence of a limit cycle surrounding either U_L or U_R.

To fix ideas, assume that (U_L, U_R) corresponds to a Lax 2-shock and that U_L is a saddle point and U_R an attractor surrounded by a limit cycle. The associated dynamical system is determined by the parameters (ϕ_-, μ) that belong to the limit cycle region in the universal unfolding of a Bogdanov–Takens bifurcation. See Fig. 2. For this kind of initial data U_R lies on the 2-shock branch of the Hugoniot curve through U_L, or, equivalently, U_L lies on the backward 2-shock branch through U_R. A typical structure of the Hugoniot curves is depicted in Fig. 4. This figure shows the forward 1-wave curve through U_L and the backward 2-wave curve through U_R. A point of intersection of these two curves would give a solution. However, the gap in the backward 2-wave curve, due to viscous inadmissibility, keeps

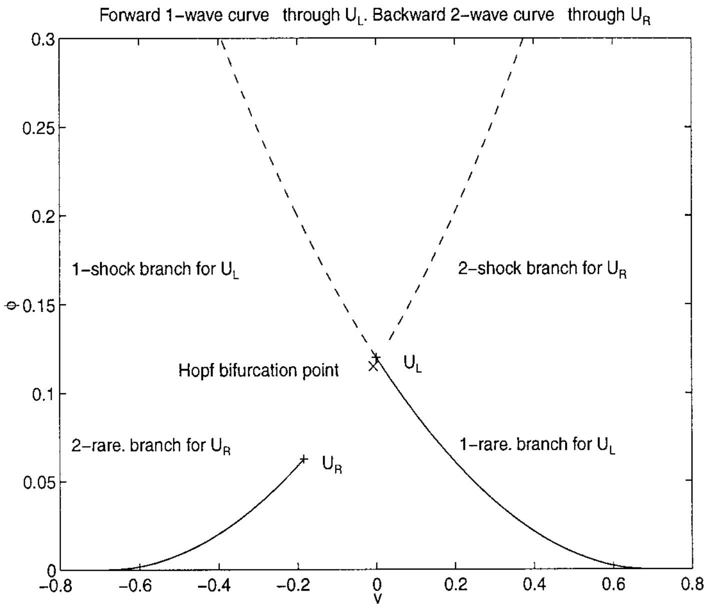

FIG. 4. The forward 1-wave curve through U_L and the backward 2-wave curve through U_R. The figure shows that these two curves do not intersect. The Hugoiot curve through U_R contains U_L. However, this part of the Hugoniot curve does not belong to the backward 2-wave curve of U_R because there is no viscous profile from U_L to U_R due to a limit cycle that was born in a nearby Hopf bifurcation point and dies at a homoclinic loop bifurcation point. The Hopf bifurcation point is also shown in this figure. The corresponding dynamical systems are determined by the parameters $\phi_- = 0.0642$ and $\mu_1 \leqslant \mu \leqslant \mu_2$, shown in Fig. 2. The dynamical system having U_L and U_R as critical points is determined by $\phi_- = 0.642$, $\mu = 0.142$. The segment of the Hugoniot curve through U_R between U_R and the Hopf bifurcation point is inadmissible because it contains the points U_* for which U_* is a saddle point and U_R a repellor. The corresponding dynamical systems are determined by $\phi_- = 0.0642$ and $0 \leqslant \mu \leqslant \mu_1$; see Fig. 2.

the curves from crossing. The part of the shock curve between the Hopf bifurcation point and the homoclinic bifurcation point, *which includes* U_L, is inadmissible due to a limit cycle in the dynamical system keeping the saddle point from connecting to the attractor at U_R. (The rest of the inadmissible points U_* fail due to the dynamical system having a *repellor* at U_R (cf. Fig. 2), which makes an admissible 2-shock from U_* to U_R impossible.)

We proceed by showing that for any U_L and U_R in the strictly hyperbolic region, the forward 1-wave curve through U_L is strictly decreasing, and the backward 2-wave curve through U_R is strictly increasing. The forward 1-wave curve consists of the viscous admissible portion of the 1-shock branch and the 1-rarefaction branch. The 1-shock branch through U_L (30) is decreasing because

$$\frac{dv}{d\phi} = -\frac{\sqrt{2}/2\phi + 3\sqrt{2}/2\phi_L}{(\phi+\phi_L)^{3/2}} < 0$$

and, similarly, it is obvious from (29) that the 1-rarefaction branch is decreasing. The backward 2-shock branch through U_R satisfies

$$\frac{dv}{d\phi} = \frac{\sqrt{2}/2\phi + 3\sqrt{2}/2\phi_R}{(\phi+\phi_R)^{3/2}} > 0$$

and it is obvious from (31) that the 2-rarefaction branch given by (31) through U_R is increasing as well.

Therefore, if, for example, (U_L, U_R) corresponds to a Lax 2-shock that does not admit a viscous profile due to a limit cycle surrounding one of the critical points (the attractor), U_L lies on the backward 2-shock branch through U_R that consists of points that are not viscous admissible. Thus this portion of the shock curve *does not* belong to the backward 2-wave curve through U_R. Since the 1-shock branch and the 1-rarefaction branch are decreasing, and since the backward 2-shock branch and the backward 2-rarefaction branch are increasing, the two wave curves never meet.

Note that we could repeat the same reasoning for the situation when U_L and U_R correspond to a viscous inadmissible Lax 1-shock wave. The same conclusion holds. Thus, by using the fact that the wave curves are monotonic, we have proved the following theorem.

THEOREM 3. *Let Riemann initial data* (2) *be such that the left state* U_L *and the right state* U_R *form a Lax admissible shock wave which does not satisfy the viscous profile entropy criterion due to the presence of a limit cycle surrounding either* U_L *or* U_R. *Then the Riemann problem for the* *shallow water equations* (1) *does not have a scale-invariant solution in which all shock waves admit viscous profiles with a viscosity matrix of the form* (5).

Note that we have proved that whenever there is a gap in the wave curves due to shock inadmissibility, there will be nonexistence of solutions in which all shock waves comply with the viscous profile entropy criterion. In Section 3 we have shown that such gaps in the wave curves occur naturally if a nondiagonal, positive definite, symmetric viscosity matrix is considered. A study of what kind of solution exists in those situations is presented in [17]. The results of that work show that measure-valued solutions exhibiting continuously generated oscillations take place.

5. CONCLUSIONS

In this paper we have shown that for a generic, positive definite, symmetric viscosity matrix and an open set of hyperbolic Riemann initial data, the shallow water equations do not admit a "classical weak solution" in which all shock waves comply with the viscous profile entropy criterion. The presence of such initial data is tied with the presence of a nontrivial region of hyperbolic states that are linearly unstable in the sense studied by Majda and Pego. At the boundary of Majda–Pego stable points, a Bogdanov–Takens bifurcation takes place. The initial data that leads to nonexistence can be found in the universal unfolding of the Bogdanov–Takens bifurcation. Such initial data correspond to the singularities in the dynamical systems that are determined by the parameters in the limit cycle region of the universal unfolding of the Bogdanov–Takens bifurcation. The limit cycle region lies between the Hopf and the homoclinic bifurcation loci. We have shown that at least part of the initial data associated with the limit cycle region (in a neighborhood of the Hopf locus) *is Majda–Pego unstable*. It is not clear whether, in general, both initial states U_L and U_R associated with the limit cycle region are stable or not. However, we have shown that in the example closely analyzed in this manuscript (6), both initial states, U_L and U_R, are Majda–Pego unstable.

A natural question to ask at this point is: "What kind of a solution exists in those situations when a classical weak solution does not exist?" The answer to this question is given in [17]. We show, among other things, that the Riemann problem with the data specified in (6) has a solution which exhibits continuously generated oscillations that satisfy the system of conservation laws in a measure-valued sense. More precisely, in [17] we investigate solutions of the inviscid conservation law (1), with initial data specified in (6), by considering the limit of the associated parabolic problems

(7) with the viscosity matrix given in (6). For each fixed ε we calculate solutions of the parabolic problems numerically. We observe that these solutions exhibit oscillations that are uniformly bounded in amplitude, whose frequency increases as $\varepsilon \to 0$. We show that the oscillatory solutions converge, in the weak-* topology of L^∞, and we prove that the limit, as $\varepsilon \to 0$, is a measure-valued solution of the inviscid system of conservation laws.

6. APPENDIX

The coefficients a_{ij}, b_{ij}, α_i, and β_i that are used in Section 3 and first appear in Eq. (20) are given by the following (here α and β are given by (17) and (18) and ϕ^*_- by (16)):

$$a_{11} = -c + \frac{\alpha}{\beta} b,$$

$$a_{12} = -\frac{b}{\alpha},$$

$$a_{21} = \frac{\alpha}{\beta^2} (-\beta\alpha c - b\beta^2 + \alpha^2 b + a\alpha\beta)$$

$$a_{22} = -\frac{\alpha b + a\beta}{\beta}$$

$$b_{11} = \frac{1}{2\sqrt{\phi^*_-}} a_{11} + b,$$

$$b_{12} = \frac{1}{2\sqrt{\phi^*_-}} a_{12},$$

$$b_{21} = \frac{1}{2\sqrt{\phi^*_-}} a_{21} + \frac{\alpha}{\beta} (\alpha b + a\beta),$$

$$b_{22} = \frac{1}{2\sqrt{\phi^*_-}} a_{22},$$

$$\alpha_1 = \frac{c\alpha\beta - 2\alpha^2 b}{2\beta},$$

$$\alpha_2 = \frac{\alpha}{2\beta^2} (c\beta\alpha^2 - 2b\alpha^3 + \beta^2\alpha b - 2a\alpha^2\beta),$$

$$\beta_1 = b,$$

$$\beta_2 = \frac{\alpha}{2\beta^2} (2\alpha\beta^2 + 2\alpha\beta b).$$

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